RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 53, NUMBER 2 FEBRUARY 1996

Stabilization of soliton instabilities by higher-order dispersion: Fourth-order nonlinear Schrödinger-type equations

V. I. Karpman*

Center for Plasma Physics and Nonlinear Dynamics, Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel

(Received 27 November 1995)

Stability of the soliton solutions to the fourth-order nonlinear Schrödinger equations with arbitrary power nonlinearities in different space dimensions is studied. Necessary and sufficient conditions of the stability with respect to small perturbations are found. The results obtained represent also necessary conditions of the stabilization of self-focusing and collapse by high-order dispersion.

PACS number(s): 42.65.Tg, 42.65.Jx, 52.35.Mw

The problem of soliton stability, considered in this paper, is closely related to the collapse (blowup) of nonlinear waves, one of the basic phenomena in nonlinear physics. The instabilities, leading to the collapse, depend on the number of space dimensions and strength of nonlinearity [1,2]. The progress in their study is greatly stimulated by considering comparatively simple models, convenient for analytical and numerical investigations. In this way many important properties of instabilities, leading to the collapse, have been discovered. Up to now, the main attention in the studies of the collapse stabilization has been paid to the nonlinearity saturation (e.g., [3]). On the other hand, effects of higher-order dispersion may also play a significant role [4]. Here, we report the results of a study of the stabilization of soliton instabilities, with respect to small perturbations, in the model based on the fourth-order nonlinear Schrödinger (NLS) equation

$$i\partial_t\Psi+\tfrac{1}{2}\Delta\Psi+\frac{\gamma}{2}\Delta^2\Psi+f(|\Psi|^2)\Psi=0,\quad f(u)=u^p \quad (1)$$

where

$$p\geqslant 1,\quad \Delta=\nabla_\alpha^2\equiv\nabla_\alpha\nabla_\alpha,\quad \alpha=1,\ldots,D,\quad D=1,2,3,$$

(summation over repeated indices is assumed; D is the number of space dimensions).

At $\gamma=0$, Eq. (1) has been considered in a large variety of papers (see, e.g., Refs. [1,2] and references therein). One of the main results achieved is that the solitary wave solutions of Eq. (1) with $\gamma=0$ (which are localized pulses or wave beams) are unstable at $p\geqslant 2$ ($D=1$) and $p\geqslant 1$ ($D\geqslant 2$). These instabilities may result in a blowup at finite t (self-focusing or collapse) [1,2].

At $\gamma\neq 0$, and $D=2$, $p=1$ Eq. (1) has been considered in Refs. [4,5] in a study of the self-focusing in high dispersive systems. It has been shown by qualitative reasoning [4] and numerical investigations [5] that the fourth-order derivative term plays a very important role, depending on sgnγ. At $\gamma>0$, it leads to radiation and consequent defocusing of a wave beam; at $\gamma<0$, it makes possible the formation of stable stationary wave beams. One dimensional equations (1) with $\gamma\neq 0$ and $p=1$ have been considered in connection with the nonlinear fiber optics [6,7] and the theory of optical solitons in gyrotropic media [8]. A straightforward analytical approach [6] leads to results similar to those obtained earlier qualitatively and numerically for the wave beams [4,5]: at $\gamma>0$ there are radiating solitons while at $\gamma<0$ there can exist stationary solitons with monotonic or oscillatory asymptotics (their existence has been confirmed numerically [9]). Along with that, at $D=1$, $p>1$, and $\gamma>0$ there are no stable solitons similar to the case $\gamma>0$, $p\geqslant 4$ in high dispersive Korteweg–de Vries type equations [10].

Here, we study the soliton stability at $\gamma<0$ and any $D,p\geqslant 1$ satisfying the relation $(2-D)p+2>0$ [it seems that for other D and p the soliton solutions to Eq. (1) do not exist]. Based on two conjectures which are supported by numerical results, we derive necessary and sufficient conditions of the soliton stability with respect to small perturbations.

Equation (1) can be obtained from the action principle. From the symmetries of the Lagrangian, there follow conserved quantities [4]. Here we shall use two of them: the number of quanta and energy

$$N=\int d^D\mathbf{x}|\Psi|^2,$$

$$H=\int d^D\mathbf{x}\left[\tfrac{1}{2}\nabla_\alpha\Psi\nabla_\alpha\Psi^*-\frac{\gamma}{2}\Delta\Psi\Delta\Psi^*-F(|\Psi|^2)\right], \quad (2)$$

where $F'(u)=f(u)$. The corresponding fluxes are given in Ref. [4].

The soliton solutions of Eq. (1) have the form $\Psi_s(\mathbf{x},t)=\Phi_s(\mathbf{x})\exp(i\Lambda t)$. Substituting this in (1) we come to the soliton equation

$$\gamma\Delta^2\Phi_s+\Delta\Phi_s+2[f(\Phi_s^2)-\Lambda]\Phi_s=0. \quad (3)$$

It can be obtained from the constrained variational problem

$$\delta(H+\Lambda N)=0, \quad (4)$$

i.e., the soliton is a stationary point (in functional space) of the Hamiltonian H at constant N with Λ playing the role of Lagrange multiplier.

Introducing the notations

*Electronic address: karpman@vms.huji.ac.il

$$J_1=\int d^D\mathbf{x}(\nabla\Phi_s)^2,\quad J_2=\tfrac{1}{2}\int d^D\mathbf{x}(\Delta\Phi_s)^2,$$

$$I_n=\int d^D\mathbf{x}\ \Phi_s^{2n} \tag{5}$$

we have

$$H_s\equiv H\{\Phi_s\}=\tfrac{1}{2}J_1-\gamma J_2-\frac{1}{p+1}I_{p+1}. \tag{6}$$

Multiplying Eq. (3) by Φ_s and integrating by parts, we obtain

$$\gamma J_2-\tfrac{1}{2}J_1+I_{p+1}-\Lambda N_s=0,\quad N_s=N\{\Phi_s\}. \tag{7}$$

Another equation, connecting integrals (5), can be obtained by means of the scaling transformation $\mathbf{x}'=\alpha\mathbf{x}$ [11,1] which gives, together with (7), a system of two linear algebraic equations for the integrals (5). Solving this system, we have

$$J_1=\frac{2pD}{(2-D)p+2}\Lambda N_s+2\frac{(4-D)p+4}{(2-D)p+2}\gamma J_2, \tag{8}$$

$$I_{p+1}=\frac{2(p+1)}{(2-D)p+2}\Lambda N_s+\frac{2(p+1)}{(2-D)p+2}\gamma J_2. \tag{9}$$

Substituting (8) and (9) into (6), we obtain

$$H_s=\frac{pD-2}{(2-D)p+2}\Lambda N_s+\frac{2p}{(2-D)p+2}\gamma J_2. \tag{10}$$

At $\gamma=0$, we come to the expression obtained in Refs. [12,1].

Now, consider the scaling transformation conserving $N\{\Psi\}$

$$\Psi'=\lambda^{D/2}\Psi(\lambda\mathbf{x},t). \tag{11}$$

Then H_s is transformed to $H(\lambda)$,

$$H(\lambda)=\tfrac{1}{2}\lambda^2J_1-\lambda^4\gamma J_2-\frac{\lambda^{Dp}}{p+1}I_{p+1}. \tag{12}$$

Substituting here (8) and (9), we have

$$H(\lambda)=\frac{\Lambda N_s}{(2-D)p+2}(pD\lambda^2-2\lambda^{Dp})+\frac{\gamma J_2}{(2-D)p+2}\{[(4-D)p+4]\lambda^2-[(2-D)p+2]\lambda^4-2\lambda^{Dp}\}. \tag{13}$$

Evidently, $H(1)=H_s$ and $H'(1)=0$.

Let us now consider the soliton stability with respect to small perturbations. Assume that

$$\Psi=(\Phi_s+\phi)\exp(i\Lambda t),|\quad |\phi|\ll 1,\quad (\Phi_s,\phi)=0 \tag{14}$$

where $\phi=v+iw$ and $(F,G)=\int d^D\mathbf{x}\,F^*G$. From (14) it follows that, to the first order, $N\{\Psi\}=N_s$. Substituting (14) into (1) and neglecting second-order terms, we come to the equations

$$\partial_t v=\hat{L}w,\quad \partial_t w=-\hat{M}v,\quad (\Phi_s,v)=(\Phi_s,w)=0, \tag{15}$$

where we introduced the Hermitian operators

$$\hat{L}=-\tfrac{1}{2}[\gamma\Delta^2+\Delta+2f(\Phi_s^2)-2\Lambda],$$

$$\hat{M}=\hat{L}-2f'(\Phi_s^2)\Phi_s^2. \tag{16}$$

It is easy to check that

$$\hat{L}\Phi_s=0,\quad \hat{M}\nabla\Phi_s=0. \tag{17}$$

Thus, Φ_s and $\nabla\Phi_s$ are eigenfunctions of operators $\hat{L}$ and $\hat{M}$, respectively, with eigenvalues equal to zero. Formally, this conclusion is the same as for $\gamma=0$ [1].

Assuming that v and w depend on time as $\exp(-i\Omega t)$, we have

$$\Omega^2=\frac{(w,\hat{L}w)}{(w,\hat{M}^{-1}w)}. \tag{18}$$

From (18) it follows that the soliton is stable if both operators, $\hat{L}$ and $\hat{M}$, are positive or negative definite in the subspace, orthogonal to Φ_s.

Let us first investigate the operator $\hat{L}$. We know one of its eigenfunctions, Φ_s. For $\gamma=0$, it corresponds to the ground state because it has no nodes. Then the lowest eigenvalue is zero and $\hat{L}$ is positive definite in the subspace, orthogonal to Φ_s. For $\gamma<0$, the situation is more complicated. Indeed, performing the scaling

$$\Phi_s=(2\Lambda)^{1/2p}\phi_s,\quad x=(2\Lambda)^{-1/2}\xi, \tag{19}$$

we transform Eq. (4) to the form

$$\varepsilon\Delta^2\phi_s-\Delta\phi_s-(2\phi_s^{2p}-1)\phi_s=0,\quad \varepsilon=-2\Lambda\gamma, \tag{20}$$

where now $\Delta=\partial^2/\xi_\alpha^2$ (and $\varepsilon>0$). Thus, at $|\xi|\gg 1$ [6]

$$\phi_s\propto\exp(-\kappa|\xi|),\quad \kappa^2=\frac{1}{2\varepsilon}(1\pm\sqrt{1-4\varepsilon}). \tag{21}$$

This is valid for any number of dimensions D. From (21) it is seen that the asymptotic behavior of the soliton is monotonic or oscillating at, respectively, $\varepsilon\leq 1/4$ or $\varepsilon>1/4$. In the last case, Φ_s has nodes at $|\xi|\gg 1$. Nevertheless, numerical investigations at $D=1,2,3$ and sufficiently small ε (which, however, can be even larger than 1/4) show that Φ_s remains to be the eigenfunction of the ground state for $\gamma<0$. This gives us a reason for the following conjecture: there exists $\varepsilon_p>0$, such that at $0<\varepsilon<\varepsilon_p$, the operator $\hat{L}$ is positive definite in the subspace, orthogonal to Φ_s. Generally, we cannot exclude that $\varepsilon_p\gg 1$ (see also the end of this paper).

If, and where, this conjecture is valid, the necessary and sufficient condition of the soliton stability is the positive definiteness of operator $\hat{M}$ in the subspace, orthogonal to Φ_s. Let us find when it is possible.

Acting as in Ref. [12], we differentiate Eq. (4) with respect to Λ. This gives

$$\hat{M}\left(\frac{\partial\Phi_s}{\partial\Lambda}\right)_\lambda=-\Phi_s,\quad \text{i.e.,}\quad \left(\frac{\partial\Phi_s}{\partial\Lambda}\right)_\gamma=-\hat{M}^{-1}\Phi_s. \quad (22)$$

From the second of Eqs. (17) it follows that

$$(\Phi_s,\phi_0)=0 \quad \text{where} \quad \hat{M}\phi_0=0 \quad (23)$$

($\phi_0=\text{grad}\Phi_s$). This is in agreement with (22). Consider now the extremum of quadratic form $(\phi,\hat{M}\phi)$ with the constraints

$$(\phi,\phi)=1, \quad (\Phi_s,\phi)=0. \quad (24)$$

Using the Lagrange multipliers μ and C, we come to the variational problem

$$\delta[(\phi,\hat{M}\phi)-\mu(\phi,\phi)-C(\phi,\Phi_s)]=0 \quad (25)$$

which leads to the equation

$$(\hat{M}-\mu)\phi=C\Phi_s. \quad (26)$$

From (24) it is seen that μ is an extremum of $(\phi,\hat{M}\phi)$. Thus, to find the signature of $(\phi,\hat{M}\phi)$ we must solve Eq. (26) and find minμ. This can be done as in Ref. [12]. We write

$$\hat{M}\phi_n=\mu_n\phi_n, \quad \Phi_s=\sum_{n\neq 0} b_n\phi_n \quad (27)$$

[the term with $n=0$, corresponding to the eigenvalue equal to zero, is absent in the last equation due to (23)]. Then

$$\phi=C\sum_{n\neq 0}\frac{b_n\phi_n}{\mu_n-\mu}. \quad (28)$$

Assuming that ϕ satisfies the second of constraints (24), we obtain the following equation for μ:

$$r(\mu)\equiv\sum_{n\neq 0}\frac{|b_n|^2}{\mu_n-\mu}=0, \quad (29)$$

i.e., all extremal values μ of the quadratic form $(\phi,\hat{M}\phi)$, with the constraints (24), are the roots of the function $r(\mu)$. We have to investigate the sign of the lowest of them. It is easy to find from (29) that min$\mu>0$ if $\hat{M}$ has only one negative eigenvalue and, in addition, $r(0)<0$. Then, from

$$r(0)=(\Phi_s,\hat{M}^{-1}\Phi_s)=-\frac{1}{2}\left(\frac{\partial N_s}{\partial\Lambda}\right)_\gamma \quad (30)$$

it follows that the quadratic form $(\phi,\hat{M}\phi)$ is positive in the subspace, orthogonal to Φ_s, if

$$(\partial N_s/\partial\Lambda)_\gamma>0. \quad (31)$$

Numerical calculations in some particular cases show that operator $\hat{M}$ has indeed *only one negative eigenvalue at least for small* ε (for $\varepsilon=0$ this is shown in [1,12]). We adopt this as the second conjecture. Then, relation (31) is the necessary and sufficient condition of

$$(\phi,\hat{M}\phi)>0 \quad (32)$$

at $(\Phi_s,\phi)=0$ and sufficiently small ε. From this it follows that (31) is the necessary and sufficient condition of the soliton stability with respect to small perturbations. Using (19) and defining

$$i_n=\int d^D\xi|\phi_s|^{2n}, \quad j_2=\tfrac{1}{2}\int d^D\xi|\Delta\phi_s|^2, \quad (33)$$

we have $N_s=\Lambda^{(2-Dp)/2p}i_1(\varepsilon)$, i.e.,

$$\left(\frac{\partial N_s}{\partial\Lambda}\right)_\gamma=\Lambda^{(2-Dp-2p)/2p}\left(\frac{2-Dp}{2p}i_1+\varepsilon\frac{\partial i_1}{\partial\varepsilon}\right). \quad (34)$$

Thus condition (31) is reduced to

$$(Dp-2)i_1<2p\varepsilon\,\partial i_1/\partial\varepsilon. \quad (35)$$

At $\gamma=0$, the right hand side of (35) vanishes and the stability condition is $p<2/D$; in the "critical" case $p=2/D$ and $\gamma=0$, the solitons are unstable [1,12].

Numerical experiments for particular cases suggest that $i_1(\varepsilon)$ is an increasing function (at sufficiently small ε). If this will be confirmed, the solitons are stable at

$$p\leq 2/D \quad (\gamma<0). \quad (36)$$

At $D=2,1$ this was checked by direct numerical experiments [5,13]. Condition (35) can also be used at $p>2/D$, to determine the values of ε (at fixed p and D) for which solitons are stable. This will be done in a separate paper.

From the above analysis it follows that the fourth derivative term in Eq. (1) at $\gamma<0$ plays a stabilizing role. This can be seen also from (3) because at $\gamma<0$ the second term in H is positive, like the first one, while the third term, promoting the collapse instability, is negative.

Now, let us derive other conditions of the soliton stability. From (3) we have

$$[\delta^2(H+\Lambda N)]_s=\int d^D\mathbf{x}(\phi^*,M\phi)-\tfrac{1}{2}\int d^D\mathbf{x}f'(\Phi_s^2)\Phi_s^2(\phi-\phi^*)^2. \quad (37)$$

Evidently, $(\phi-\phi^*)^2<0$. Then, taking into account (4) and (32), we come to the following necessary condition of the soliton stability:

$$H_s+\Lambda N_s=\min(H+\Lambda N). \quad (38)$$

On the other hand, if (38) holds, $H+\Lambda N$ can be considered as a Lyapunov functional and therefore (38) is also a sufficient condition of the soliton stability.

From (38) it follows that if the soliton is stable, $H(\lambda)$, defined in (13), must have a minimum at $\lambda=1$. [We recall that transformation (11) conserves $N\{\Psi\}$.] Then $H''(1)$ must be positive. Defining

$$R(\varepsilon;p,D)=\frac{|\gamma|J_2}{\Lambda N_s}=\frac{\varepsilon j_2(\varepsilon)}{i_1(\varepsilon)},$$

$$R_{\rm cr}(p,D)\equiv\frac{(Dp-2)Dp}{(Dp)^2-6Dp+8(p+1)},\qquad(39)$$

where j_2 is given by (33), we have

$$H''(1)=2\frac{(Dp)^2-6Dp+8(p+1)}{(2-D)p+2}\Lambda N_s(R-R_{\rm cr}).\qquad(40)$$

As far as we consider only integer and positive p,

$$(Dp)^2-6Dp+8(p+1)=(Dp-3)^2+8p-1>0.$$

Also, $(2-D)p+2>0$ if

$$D=1,2,\quad p<\infty\quad\text{and}\quad D=3,\quad p=1.\qquad(41)$$

Thus, at (41) and $\gamma<0$, we come to the following necessary condition of the soliton stability:

$$R(\varepsilon;p,D)>R_{\rm cr}(p,D),\qquad(42)$$

which is a restriction on ε. At the additional assumption (36), condition (42) is fulfilled automatically because $R_{\rm cr}(p,D)<0$.

There is some evidence that at $p\geqslant2$, $D=3$ the soliton solutions to Eq. (1) do not exist. [Note that the denominator in (40) vanishes at $p=2$, $D=3$.] We shall not discuss this case anymore.

Thus, adopting the two above conjectures, we come to the necessary and sufficient conditions of the soliton stability with respect to small perturbations. We have also found that the fourth-order dispersive term in NLS-type equations (1) stabilizes instabilities which may finally evolve into collapselike processes as it takes place, e.g., in the case $p=D/2$, $\gamma=0$. The soliton stability with respect to finite perturbations will be considered in a separate work.

This work was partially supported by the Israel Ministry of Science and the Arts.

[1] E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, Phys. Rep. **142**, 103 (1986); J. J. Rasmussen and K. Rypdal, Phys. Scr. **33**, 481 (1986).
[2] Wave Collapses, Proceedings of the International Workshop on Wave Collapse Physics, edited by V. E. Zakharov [Physica (Amsterdam) D **52**, 1 (1991)]; S. K. Turitsyn, Phys. Rev. E **47**, R13 (1993).
[3] E. W. Laedke, K. H. Spatschek, and L. Stenflo, J. Math. Phys. **24**, 2764 (1983); E. W. Laedke and K. H. Spatschek, Phys. Rev. A **30**, 3279 (1984).
[4] V. I. Karpman, Phys. Lett. A **160**, 531 (1991).
[5] V. I. Karpman and A. G. Shagalov, Phys. Lett. A **160**, 538 (1991).
[6] V. I. Karpman, Phys. Lett. A **193**, 355 (1994).
[7] M. Karlsson and A. Höök, Opt. Commun. **104**, 303 (1994); N. N. Akhmediev, A. V. Buryak, M. Karlsson, *ibid.* **110**, 540 (1994).
[8] V. I. Karpman, Phys. Rev. Lett. **74**, 2455 (1995).
[9] V. I. Karpman and A. G. Shagalov (unpublished).
[10] V. I. Karpman, Phys. Lett. A **186**, 303 (1994).
[11] G. H. Derrick, J. Math. Phys. **5**, 1252 (1964).
[12] N. G. Vakhitov and A. A. Kolokolov, Izv. Vuz. Radiofiz. **16**, 1020 (1973) [Radiophys. and Quantum Electronics **16**, 783 (1975)].
[13] V. I. Karpman *et al.* (unpublished).